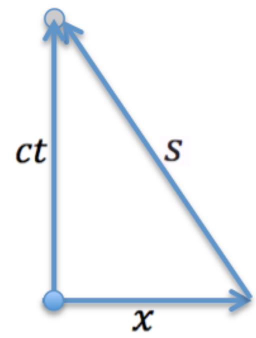
Derive the Equation for Relativistic Energy in Minkowski Space-Time



A particle moves a distance x through space in a time interval t. During the same time interval, it moves through the time dimension at the speed of light, travelling a distance of ct. The total distance travelled in space-time is given by the vector s. In Minkowski space-time, this invariant interval is:

$$\boldsymbol{s^2 = (ct)^2 - x^2}$$

Because the speed of light in space-time is constant, we must **subtract** the space distances from the time difference.

We can replace the space component x in with its velocity multiplied by time ($x = vt$), to express the space-time interval as a factor of time:

$$s^2 = (ct)^2 - (\boldsymbol{vt})^2 = t^2 \cdot (c^2 - v^2)$$

Now express the space-time interval as a factor of the distance travelled through time (ct):

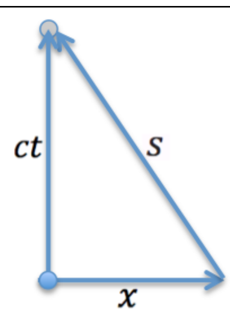
$$s^2 = t^2 \cdot \frac{\boldsymbol{c^2}}{\boldsymbol{c^2}} \cdot (c^2 - v^2) = \boldsymbol{c^2} \cdot t^2 \cdot \frac{(c^2 - v^2)}{\boldsymbol{c^2}} = (ct)^2 \cdot \left(1 - \frac{v^2}{c^2}\right)$$

Take the square root to express the space-time interval in terms of the time vector (ct) divided by the Lorentz factor:

$$s = c \cdot t \cdot \sqrt{1 - \frac{v^2}{c^2}} = \frac{\boldsymbol{c \cdot t}}{\boldsymbol{\gamma}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

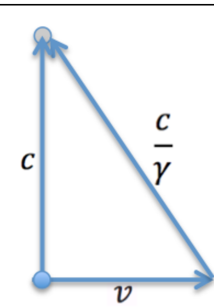
Divide the three distances by time to get the velocity relationship in space-time. Then multiply all three sides by gamma, to make the invariant velocity through space-time equal to the speed of light. In any particular frame of reference, velocity must be expressed as the three space components plus one time component.

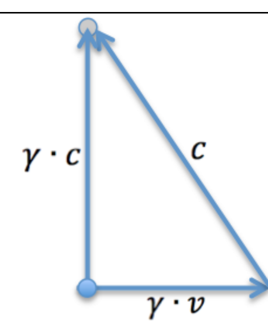


$$\frac{s}{t} = \frac{\boldsymbol{c \cdot t}}{\boldsymbol{\gamma}} \cdot \frac{1}{t} = \frac{c}{\gamma}$$

$$\frac{ct}{t} = c$$

$$\frac{x}{t} = v$$





If our particle moving through space-time has a mass, we can express its momentum in space-time by multiplying the velocity in each dimension through by its mass.

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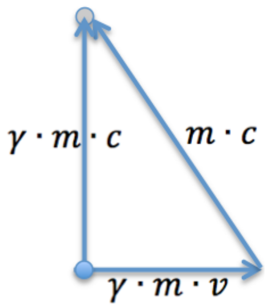
<td>
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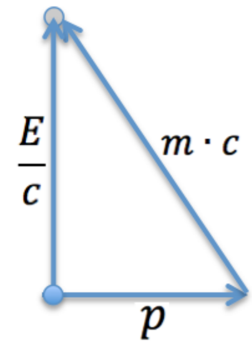
<td>Momentum in time:

$$\gamma \cdot m \cdot c = \frac{\gamma \cdot m \cdot c^2}{c} = \frac{E}{c}$$

Relativistic Momentum in space:

$$\gamma \cdot m \cdot v = p$$</td>
<td>
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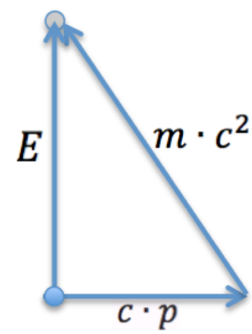
[Unfortunately, at this point we are going to assume what we are supposedly trying to prove.] If the energy of a particle is its momentum moving through time at the speed of light, then the Total Relativistic Energy in the time dimension is ($E = \gamma \cdot m \cdot c^2$). Thus energy and momentum are linked in space-time, with energy in the time dimension, and the relativistic momentum (p) in the three space dimensions.

To solve for the total energy, multiply everything through by the speed of light, c. It follows from the Minkowski metric that:

$$(m \cdot c)^2 = E^2 - (c \cdot p)^2$$



We can see that momentum flows in the space dimension, while the total energy flows through the time dimension. The difference between them is the invariant quantity of the energy of the mass itself moving through space-time.

Now solve for total amount of Energy to get the full form of the relativity equation:

$$\boldsymbol{E^2 = (c \cdot p)^2 + (m \cdot c^2)^2}$$

If the particle is not in motion, it has no momentum, and this equation reduces to its more famous form:

$$\boldsymbol{E = m \cdot c^2}$$

If we have a particle in motion that has no mass (such as a photon), we are left with an equation that tells us that it must move at the speed of light, and that despite having no mass it still has momentum:

$$\boldsymbol{E = c \cdot p}$$