Four Methods to Calculate Gravitational Time Dilation

Here are four different ways to calculate the time dilation caused by a uniform gravitational field. They all arrive at the same approximation of the scale factor:

$$1 + \frac{g \cdot h}{c^2}$$

Here are the four approaches:

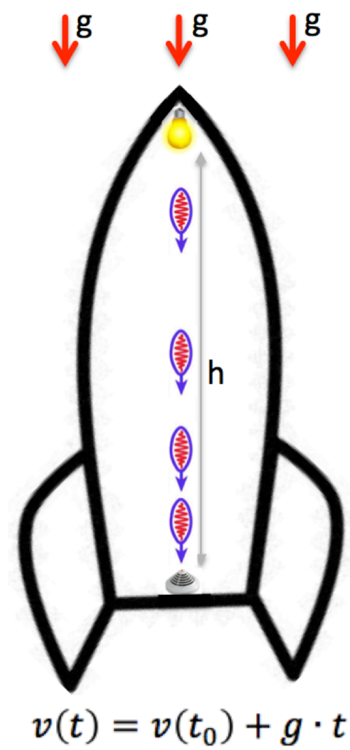
1. The first method is the simplest, relying on the basic equations of motion and the Doppler effect to calculate how quickly photons will arrive at the bottom of a gravitational field.

2. The second method was adapted from the presentation by Cristian, which uses gravitational potential energy. I start by converting the equation for gravitational potential energy to use the initial energy of the photon, rather than rely on the problematic concept of its mass.

3. The third method starts with an object on the surface of the Earth, subject to its gravitational field. It sets the gravitational potential energy equal to the kinetic energy, using the escape velocity. It then applies the standard Lorentz Transform to that velocity.

4. The fourth section is a summary of calculating uniform acceleration using a Minkowski space-time diagram. The scale factor appears in the equation of the accelerated trajectory. I can make a full calculation available if anybody want to see it.

Time Dilation from the Doppler Effect



The *equivalence principle* in general relativity tells us that acceleration is the equivalent to experiencing a uniform gravitational field that exists throughout the universe, coming from the direction of motion, for the time period of the acceleration.

A rocket with a height (h) accelerating upward at (g) is equivalent to being stationary in a uniform downward gravitational field. Light is regularly pulsing from a source at the top, and travelling to the bottom. The time the light takes to reach the bottom (when not accelerating) is:

$$t = \frac{h}{c}$$

The additional velocity (assuming $v_0 = 0$) of the accelerating rocket after time t is:

$$v_1 = v_0 + g \cdot t = g \cdot \frac{h}{c}$$

The bottom of the rocket is moving toward the oncoming light pulses, so they will get closer together, meaning the frequency will increase. The frequency represents the speed of the clock. We are observing the frequency of the clock at the top of the gravitational field, which seems to be running quickly. Therefore the clock at the bottom appears to be running slowly.

This frequency can be found using the Doppler equation. (It is only valid at a velocity much smaller than the speed of light, so this equation is not universally valid.)

$$f_1 = f_0 \cdot \frac{c + v_1}{c + 0} = f_0 \cdot \left(1 + \frac{v_1}{c}\right)$$

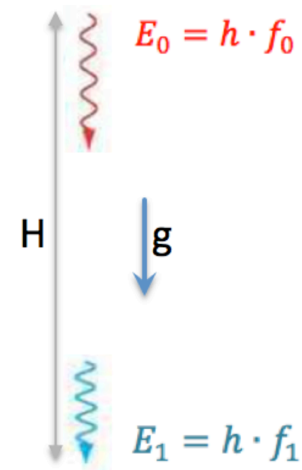
But we can calculate velocity v_1 from the acceleration and height of the rocket:

$$f_1 = f_0 \cdot \left(1 + \frac{v_1}{c}\right) = f_0 \cdot \left(1 + g \cdot \frac{h}{c} \cdot \frac{1}{c}\right)$$

Therefore, given that we measure time by the rate the light pulses arrive:

$$f_1 = f_0 \cdot \left(1 + \frac{g \cdot h}{c^2}\right) \quad \Rightarrow \quad T_f = T_0 \cdot \left(1 + \frac{g \cdot h}{c^2}\right)$$

Time Dilation from Gravitational Potential Energy



We have a photon, moving at the speed of light, at the top of a gravitational field. The initial energy of the photon is proportional (by Planck's constant h) to its frequency:

$$E_0 = h \cdot f_0$$

The photon travels to the bottom of the gravitational field, which raises its energy causing it to become blue-shifted, with a shorter wavelength and a higher frequency:

$$E_1 = h \cdot f_1$$

Gravitational potential energy (ΔE) is the difference between the energy at the top and bottom (height = H) of the gravitational field (acceleration = g). It is usually defined in terms of mass:

$$\Delta E = m \cdot g \cdot H$$

Photons do not really have mass. We can avoid problems with relativistic mass by re-writing this equation in terms of energy, based on the relationship $E = m \cdot c^2$ using the energy at the top of the field:

$$\Delta E = \frac{E_0}{c^2} \cdot g \cdot H$$

The energy at the bottom is equal to the energy at the top plus the gravitational potential energy:

$$E_1 = E_0 + \Delta E = E_0 + \frac{E_0}{c^2} \cdot g \cdot H$$

From this we calculate the ratio of the two energy values by dividing through by E_0:

$$\frac{E_1}{E_0} = \frac{E_0}{E_0} + \frac{E_0}{E_0 \cdot c^2} \cdot g \cdot H = 1 + \frac{g \cdot H}{c^2}$$

We now convert the energy into frequency. Planck's constant cancels out, so this result does not depend on quantum theory, only on the fact that energy is proportional to frequency:

$$\frac{E_1}{E_0} = \frac{h \cdot f_1}{h \cdot f_0} = 1 + \frac{g \cdot H}{c^2}$$

We can think of the frequency as how frequently the photons arrive, which are equivalent to clock ticks. Therefore the ratio of the frequencies is the factor by which time slows down in a gravitational field:

$$\frac{f_1}{f_0} = 1 + \frac{g \cdot H}{c^2}$$

Time Dilation Caused by the Earth's Gravitational Field

In general relativity, time slows down in a gravitational field. An object (mass = m) at a distance of R from the center of the Earth has a potential energy that depends on the mass of the Earth (M_E) and that distance. We can equate that to the kinetic energy of the object:

$$\frac{G \cdot M_E \cdot m}{R} = \frac{1}{2} \cdot m \cdot {v_g}^2$$

But what is this velocity, given that the motion of the object is irrelevant to this calculation? The *equivalence principle* states that there is no difference between a body experiencing a gravitational field or uniform acceleration. This velocity is called the equivalent gravitational velocity (v_g), which is the same as the escape velocity:

$$v_g = \sqrt{\frac{2 \cdot G \cdot M_E}{R}}$$

We can now calculate the time dilation at this velocity using the standard Lorentz Transformation from Special Relativity.

$$T_g = \frac{T_0}{\sqrt{1 - \frac{{v_g}^2}{c^2}}} = \frac{T_0}{\sqrt{1 - \frac{2 \cdot M_e \cdot G}{R \cdot c^2}}}$$

The acceleration at a particular radius can be found by equating the force of that acceleration to that of the universal gravitational equation:

$$m \cdot g = G \cdot \frac{m \cdot M_E}{R^2} \quad \Rightarrow \quad g = G \cdot \frac{M_E}{R^2}$$

We can therefore write the time dilation in terms of a particular uniform acceleration:

$$T_g = \frac{T_0}{\sqrt{1 - \frac{2 \cdot M_e \cdot G}{R \cdot c^2}}} = \frac{T_0}{\sqrt{1 - \frac{2 \cdot g \cdot R}{c^2}}}$$

Use a Taylor expansion to approximate the scaling factor. We can drop the exponential terms because those involve dividing by powers of the square of the speed of light:

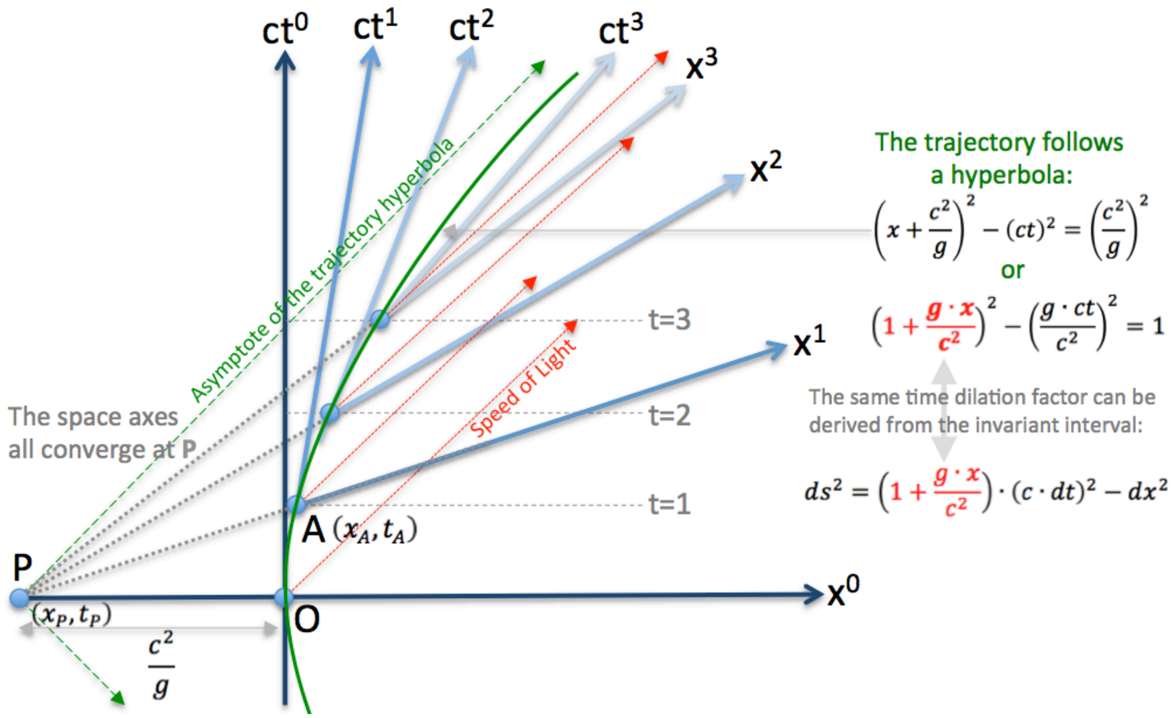
$$(1 + x)^a = 1 + \frac{a}{1!} \cdot x + \frac{a \cdot (a - 1)}{2!} \cdot x^2 + \cdots$$

$$T_g = \frac{1}{\sqrt{1 - \frac{2 \cdot g \cdot R}{c^2}}} = \left(1 + \left(-\frac{2 \cdot g \cdot R}{c^2}\right)\right)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right) \cdot \left(-\frac{2 \cdot g \cdot R}{c^2}\right) = \mathbf{1} + \frac{\boldsymbol{g \cdot R}}{\boldsymbol{c^2}}$$

Summary of Acceleration in Minkowski Space-Time

A spacecraft starts at rest at Event **O** (0,0) in the frame of reference defined by the $\boldsymbol{ct^0}$ and $\boldsymbol{x^0}$ axes. It is then subjected to a constant *proper* acceleration g, as measured in the spacecraft's moving frame of reference. A stationary observer sees the spacecraft's time, and therefore also its *co-ordinate* acceleration, slowing down.

At time $t = 1$, it has reached Event **A** and now has a velocity defined by vector ($\boldsymbol{ct^1}$). It is therefore in the frame of reference defined by the $\boldsymbol{ct^1}$ and $\boldsymbol{x^1}$ axes. As it continues to accelerate the angle between the axes becomes smaller as its velocity gets larger.



The position of each event along the world line of the accelerating spacecraft forms a hyperbola that passes through the origin. Its equation contains the time dilation factor. Note how the acceleration trajectory follows the hyperbolic shape of space-time.

At each position along the trajectory, the velocity is the tangent line to the hyperbola. The corresponding x-axis represents all the events that have the same time, in that frame of reference. Those axes all converge on Event **P**, meaning that Event **P** is always "now" along the entire trajectory. From the spacecraft point of view, time stands still at Event **P**. The asymptotes of the event horizon converge at **P**, so the space-time accessible to the spacecraft approaches zero.

Because the apparent length contraction in each frame of reference gets larger as the spacecraft moves faster, the distance between the particle at every point along the trajectory and Event **P** remains constant (in the moving frame of reference).

Derived from: http://www.personal.kent.edu/~fwilliam/Chapter%2013%20General%20Relativity.pdf
and: https://arxiv.org/pdf/physics/0601179.pdf
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