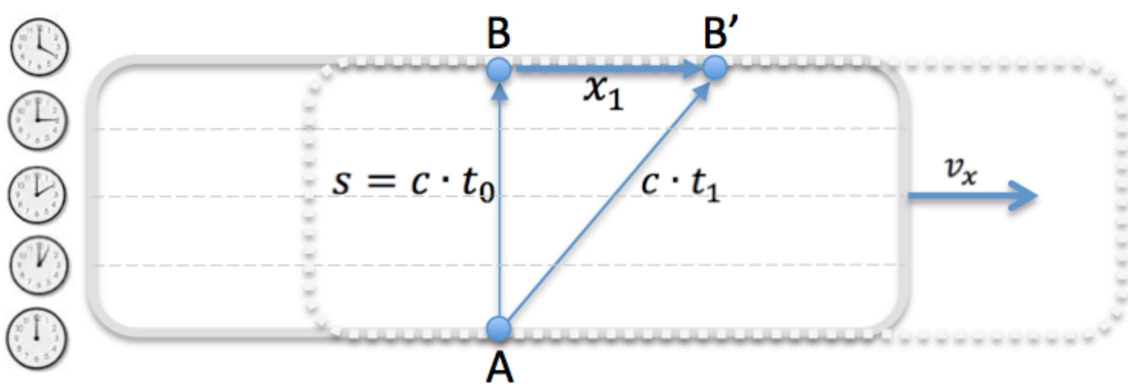
Space and Time in Special Relativity

The Speed of Light is Constant in Every Frame of Reference

Measure a Light Beam Inside the Train

Start with the perspective of a passenger on a train car (height = s). A light is flashed from its source on the floor at **A**, and reaches a detector on the ceiling above it at **B**. The light will take a certain time = t_0 to reach it. Therefore the length traversed by the light beam is: $s = c \cdot t_0$.



Measure the Same Light Beam from Outside the Moving Train

Now consider the perspective of an observer watching the train in motion toward the right at a constant velocity = v_x. The light beam starts at Event **A** as before. But it will follow a diagonal path to reach the detector at Event **B'** at the center of the car in its new position. The detector travelled a horizontal distance = x_1 from its original position at **B**.

As the speed of light must be constant in every frame of reference, from outside the train the observer must perceive it to take a longer time = t_1 to travel the longer distance from **A** to **B'**.

This is the same light beam as seen from inside the train, which means that **B** and **B'** are the same physical event. The passenger and the observer perceive them to be different. Times t_0 and t_1 are measured in different frames of reference. The Pythagoras theorem tells us that:

$$(c \cdot t_0)^2 = (c \cdot t_1)^2 - {x_1}^2$$

If we imagine the horizontal dashed lines to be clock ticks, the ticks are longer as seen from the moving frame of reference. That means we perceive that time on the train has slowed down compared to our time outside.

Calculate the Invariance of the Interval

When the train is standing still, the passenger measures the length of the light beam path as $s = c \cdot t_0$. This is the same as the height of the train. It follows that:

$$s^2 = (c \cdot t_0)^2$$

When the train is moving the observer measures the length of path of the light beam as $c \cdot t_1$. Given that it also travelled a horizontal distance of x_1, the height of the train can be calculated using the Pythagorean theorem:

$$s^2 = (c \cdot t_1)^2 - {x_1}^2$$

The value of $\boldsymbol{s}$ is the height of the train, and will be the same no matter how fast (or for any x) the train is moving. We say the height of the train is **invariant**, meaning it is the same in every frame of reference.

We can use this equation to plot all the possible combinations of events in space and time (x_i and t_i) that are the same distance apart, on a space-time diagram. It takes the form of a **hyperbola** that approaches but never reaches the speed of light. If we use y for the time direction, the equation looks like:

$$y^2 - x^2 = s^2$$

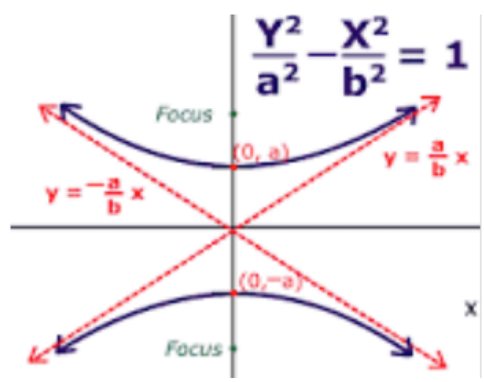
The Geometry of a Hyperbola

The general equation for a hyperbola that intersects the y-axis at $\boldsymbol{a}$, around lines with a slope of $\frac{a}{b}$ is:

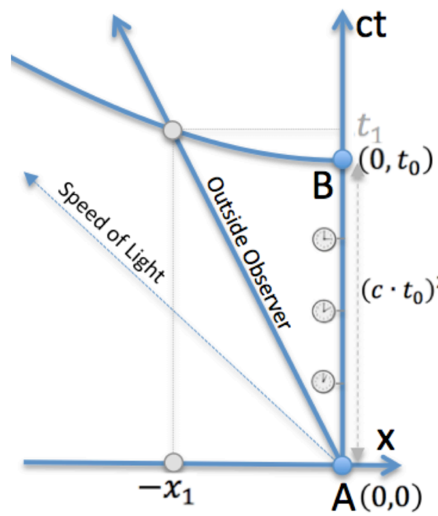
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

If $a = b$, the hyperbola will approach but never reach the 45° line, and the equation can be represented as:

$$y^2 - x^2 = a^2$$



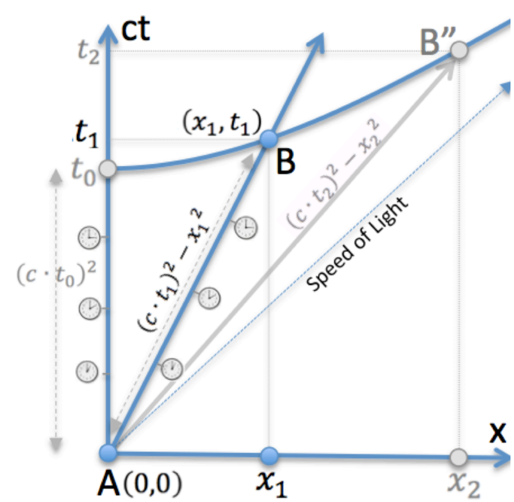
The Invariant Interval on a Space-Time Diagram



We construct a space-time diagram by choosing the perspective of one inertial frame of reference, which is considered to be standing still., while the diagram on the right is the perspective as seen by the observer on the platform.

The first diagram is from the perspective of the passenger on the train moving toward the right. The passenger sees the train as standing still in space, moving upward through time. The outside observer is moving backwards, or toward the left. The light beam follows the path from Event **A** at (0,0) to Event **B** at $(0, t_0)$, which is a distance of:

$$s^2 = (c \cdot t_0)^2$$



The outside observer sees the train moving toward the right. The light beam still follows the path from Event **A** at (0,0) to Event **B**. There is only one Event **B**, but the observer sees it in a different position, at (x_1, t_1). This space-time distance the light travelled is:

$$s^2 = (c \cdot t_1)^2 - {x_1}^2$$

This must be the same distance in space-time as seen by the train passenger. This can only be true if the observer sees time running slower on the train.

As the train goes faster, the space-time interval will be the same along the hyperbola. At Event **B"**, time on the train will be running even slower.

We must remember that these graphs describe a hyperbolic space, not a Euclidean space. A distance measured on the graph is not the same as the real distance in the underlying space. Although the space-time interval lines that join the hyperbola look like they have different lengths, they are in fact all the same length.

Calculate the Lorentz Factor from the Space-Time Interval

Now we will derive the Lorentz factor to show how time is stretched by relative motion. The train is travelling horizontally at velocity = $\boldsymbol{v_x}$ during the time $\boldsymbol{t_1}$ to cover distance $\boldsymbol{x_1}$. We can equate the space-time intervals for the stationary and moving trains:

$$(c \cdot t_0)^2 = (c \cdot t_1)^2 - \boldsymbol{x_1}^2$$

Convert the distance into velocity multiplied by time: $\boldsymbol{x_1} = \boldsymbol{v_x} \cdot \boldsymbol{t_1}$. Then substitute this back into the space-time interval:

$$(c \cdot t_0)^2 = (c \cdot t_1)^2 - (\boldsymbol{v_x} \cdot \boldsymbol{t_1})^2$$

Factor out t_1:

$$(c \cdot t_0)^2 = {t_1}^2 \cdot (c^2 - {v_x}^2)$$

Solve for t_1:

$${t_1}^2 = \frac{(c \cdot t_0)^2}{(c^2 - {v_x}^2)}$$

Find the ratio of t_1 to t_0:

$$t_1 = t_0 \cdot \sqrt{\frac{c^2}{(c^2 - {v_x}^2)}} = t_0 \cdot \frac{1}{\sqrt{\frac{c^2 - {v_x}^2}{c^2}}} = t_0 \cdot \frac{1}{\sqrt{1 - \frac{{v_x}^2}{c^2}}}$$

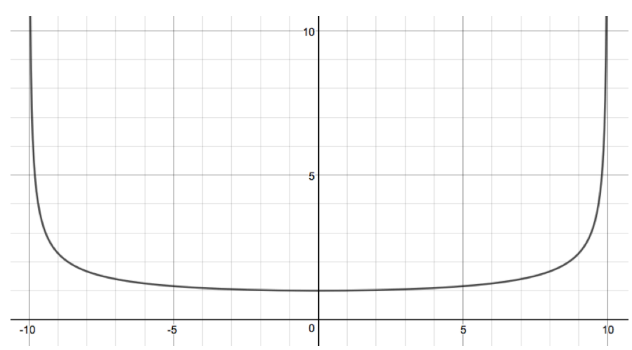
We can now express the relationship between times in different frames of reference in terms of the Lorentz Factor:

$$t_1 = t_0 \cdot \frac{\mathbf{1}}{\sqrt{\mathbf{1} - \frac{\boldsymbol{v_x}^2}{\boldsymbol{c}^2}}} = t_0 \cdot \boldsymbol{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

To show how the Lorentz Factor has little effect until you approach the speed of light, we plot the graph of the equation:

$$y = \frac{1}{\sqrt{1 - \frac{v^2}{10^2}}}$$



The velocity v is along the x-axis, while its effect on time is along the y-axis. The speed of light is arbitrarily set to 10.

The Space-time Interval is Shorter than the Time Interval

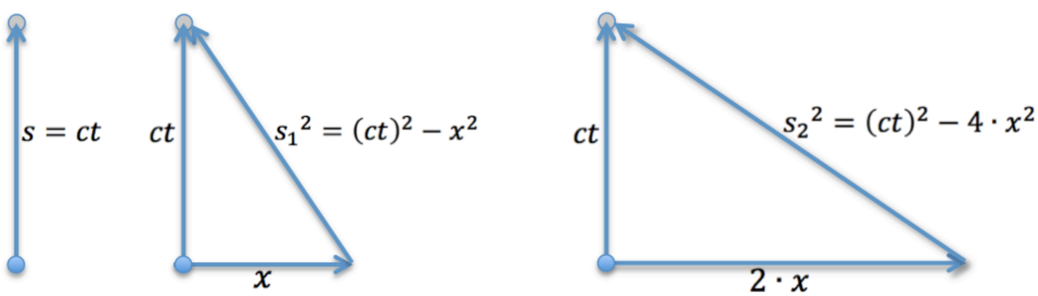
The time vector (ct) is the sum of the space vector (x) plus the space-time vector (s). That means the space-time vector must always be **shorter** than the time vector:

$$s^2 = (ct)^2 - x^2 \quad \text{or} \quad (ct)^2 = x^2 + s^2$$

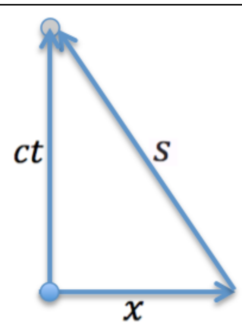
Divide through by c^2, replace the distance with $x - vt$, and express in terms of γ. We can express the space-time interval as a factor of the time interval:

$$\frac{s^2}{c^2} = \frac{(ct)^2 - (vt)^2}{c^2} = t^2 \cdot \frac{c^2 - v^2}{c^2} = t^2 \cdot \left(1 - \frac{v^2}{c^2}\right) = \frac{t^2}{\gamma^2} \quad \Rightarrow \quad \boldsymbol{s = \frac{ct}{\gamma}}$$

On the left, the observer has not moved in space, but travelled ct through the time dimension. This is the same as the space-time interval s. In the middle, a spaceship has travelled a distance x through space during the same time interval ct. The space-time interval s_1 is smaller (by a factor of γ) than before. On the right, the spaceship travels twice as far during the same time interval. That means it travelled faster. We can see that as the ship travels faster, the interval that it travels through space-time gets shorter.



Next, divide by time to get the velocity relationship then multiply all three sides by gamma. We are always moving through space-time at the invariant speed of light. We can only adjust our relative speeds through separate space and time.

	$\frac{s}{t} = \frac{c \cdot t}{\gamma \cdot t} = \frac{c}{\gamma}$ $\frac{ct}{t} = c$ $\frac{x}{t} = v$