Test Flight Problem Set

(1) Say whether the following is true or false and support your answer by a proof: $(\exists m \in N)(\exists n \in N)(3m+5n = 12)$

I will use basic algebra to demonstrate that there are no integer solutions for this statement.

First, solve the equation in terms of m: $3m = 12 - 5n$

Because we are working with natural numbers, there are only two possible values of n: 1 and 2. That means either:

$$3m = 12 - 5 \qquad\qquad \text{or} \qquad\qquad 3m = 12 - 10$$
$$3m = 7 \qquad\qquad\qquad\qquad\qquad 3m = 2$$

There are no natural numbers that satisfy either of these two results. Therefore there are no natural numbers m and n that satisfy the original equation.

The proposed statement is false, as was proven by algebraic manipulation of the equation. ■

(2) Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

Symbolic statement: $(\forall a \in \mathbb{Z})\big[\, 5 \mid \big(a + (a+1) + (a+2) + (a+3) + (a+4)\big)\big]$

This statement can be proven by simple algebra. Collect the terms:

$$a + (a+1) + (a+2) + (a+3) + (a+4) = 5a + 10$$

Extract the common factor:

$$5a + 10 = 5(a+2)$$

As $(a+2)$ is always an integer, $5(a+2)$ is by definition always an integer multiple of five.

Therefore the sum of any five consecutive integers must always be evenly divisible by five. ■

(3) Say whether the following is true or false and support your answer by a proof: For any integer n, the number n^2 +n+1 is odd.

Symbolic statement: $(\forall n \in \mathbb{Z})(\exists x \in \mathbb{Z})[n^2 + n + 1 = 2x + 1]$

I will show this is true by using a proof by contradiction.

Assume there is an even solution. Then, for all integers n, there must exist some integer p such that:

$$n^2 + n + 1 = 2p$$

Collect the n terms:

$$n^2 + n = 2p - 1$$

Factor:

$$n(n + 1) = 2p - 1$$

The left side is the product of two consecutive integers, one of which must be an even number. Therefore their product is even.

The right side is, by definition, an odd number.

As an even number cannot equal an odd number, we have shown there are no solutions for our proposed equation that assumes there is an even solution. As there are no even solutions, all possible solutions must therefore be odd.

This proof by contradiction demonstrates that all solutions to $n^2 + n + 1$ must be odd. ■

(4) Prove that every odd natural number is of one of the forms 4n + 1 or 4n + 3, where n is an integer.

Expressed symbolically:

$$(\forall x \in \mathbb{N})\big[2|(x+1) \Rightarrow (\exists n \in \mathbb{Z})[(x = 4n+1) \vee (x = 4n+3)]\big]$$

I will prove this by demonstrating that odd numbers can be divided into two mutually exclusive classes, and that each class leads to one of these solutions.

Every odd number can be expressed as $2k + 1$, for any integer k. The integer k itself can either be even or odd.

If k is even, represent it by $2n$, for some integer n. Substituting this into the definition of an odd number $(2k + 1)$, it follows that the odd number is:

$$2(2n) + 1 = 4n + 1$$

$4n + 1$ is one of the two target expressions. This shows that some odd numbers can be represented by $4n + 1$.

On the other hand, if the original k is not even, then it must be odd, expressed as $2n + 1$, for some integer n. Substituting this into the definition of an odd number $(2k + 1)$, it follows that the original odd number is:

$$2(2n + 1) + 1 = 4n + 3$$

$4n + 3$ is the other target expression. This shows that all numbers that did not fall into the first class of numbers (k is even) satisfy this alternate (k is odd) target expression.

I have demonstrated that all odd numbers can be represented as either $4n + 1$ or $4n + 3$. This proves the theorem. ■

(5) Prove that for any integer n, at least one of the integers n, n + 2, n + 4 is divisible by 3.

$$(\forall n \in \mathbb{Z})(\exists x \in \mathbb{Z})[(3x = n) \vee (3x = n + 2) \vee (3x = n + 4)]$$

I will prove this by demonstrating this follows an analogy with even and odd numbers, and then show that all possible variants lead to one of these three results.

We know that all numbers are either even or odd. This means they are either divisible by two, or they are not. Even and odd numbers are represented by $(2k)$ or $(2k + 1)$ respectively, where k is an integer.

The same applies to numbers divisible by three. They are either "even" (divisible by three), or "odd" (not divisible by three). The difference is that there are two classes of "odd" numbers. The "even" class is represented by $(3k)$, and the two "odd" classes are represented by $(3k + 1)$ and $(3k + 2)$.

I will examine each of these three classes, and show that they each lead to one of the expressions above. In each case I will substitute the class into one of the three forms in the proposition and show that it is divisible by three.

i) The "even" class $(3k)$: Substitute this into n for the case $(3x = n)$, which gives $3x = 3k$. This has an integer solution, therefore $(3x = n)$ works for this class of numbers which is divisible by three.

ii) The first "odd" class $(3k + 1)$: Substitute this into n for $(3x = n + 2)$, giving $3x = 3k + 3$. This reduces to $x = k + 1$, which clearly has an integer solution. Therefore $(3x = n + 2)$ works for this class of number which is not divisible by three.

iii) The other "odd" class $(3k + 2)$: Substitute this into n for $(3x = n + 4)$, giving $3x = 3k + 6$. This reduces to $x = k + 2$, which again clearly has an integer solution. Therefore $(3x = n + 4)$ works for this alternative class of number which is not divisible by three.

For every possible type of number in relation to its divisibility by three, each of them has been shown to be represented by $(3x = n)$, $(3x = n + 2)$ or $(3x = n + 4)$. Thus the theorem is proved. ■

(6) A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

This theorem has essentially been proven in the previous question (5). The proof is as follows:

No consecutive numbers $\{p,\ p+1\}$ will ever be prime, because in any two consecutive numbers one of them will always be even, meaning divisible by two, and thus not prime.

The question suggests there may be infinitely many pairs of "twin primes", meaning $\{p,\ p+2\}$.

A "prime triple" consists of the consecutive numbers $\{p,\ p+2,\ p+4\}$. In the question (5) on this exam, I just proved that one of these numbers must be divisible by 3. Therefore no such set of three consecutive numbers can be prime.

The only exception is the set {3,5,7}. While the first number (3) is of course divisible by 3, its factors are 1 and 3 itself, so it is by definition prime.

Therefore the only "prime triple" that contains only prime numbers is the set {3,5,7}. The theorem is therefore proved to be true. ■

(7) Prove that for any natural number n, $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n-1} - 2$

This is better represented as the summation:

$$\sum_{i=1}^{n} 2^i = 2^{n-1} - 2$$

I will prove this is true using the principle of mathematical induction.

First I must prove that it is true for a single case, the one at the beginning of the sequence. So I will show the statement is true for $n = 1$:

$$2^1 = 2^2 - 2$$

We can see that:

$$2 = 4 - 2$$

Therefore the proposition is true for $n = 1$.

Now I will use basic algebra to show that if the proposition is true for any $n = k$, then it must also be true for $n = k + 1$.

By definition, the sum of series up to term $k + 1$ is the sum of the series up to k, plus the next, or$(k + 1)$'th, term. Therefore:

$$\sum_{i=1}^{k+1} 2^i = \sum_{i=1}^{k} 2^i + 2^{k+1}$$

If the theorem is true, then the left side can be solved:

$$\sum_{i=1}^{k+1} 2^i = 2^{(k+1)-1} - 2 = 2^k - 2$$

Since we assume the theorem is true for some $n = k$, we know that

$$\sum_{i=1}^{k} 2^i = 2^{k-1} - 2$$

The right hand side is therefore:

$$\sum_{i=1}^{k} 2^i + 2^{k+1} = (2^{k-1} - 2) + 2^{k-1}$$

$$(2^{k-1} - 2) + 2^{k-1} = 2^{k-1} + 2^{k-1} - 2$$

$$2^{k-1} + 2^{k-1} - 2 = 2(2^{k-1}) - 2$$

$$2(2^{k-1}) - 2 = 2^k - 2$$

This means that:

$$\sum_{i=1}^{k} 2^i + 2^{k+1} = 2^k - 2$$

But we already showed that the left side was:

$$\sum_{i=1}^{k+1} 2^i = 2^k - 2$$

The left hand side is therefore the same as the right hand side.

We have shown the proposition is true for $n = 1$, and that if it is true for any $n = k$ then it is always true for $n = k + 1$. Therefore the theorem is true by the principle of mathematical induction. ■

(8) Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{M \cdot a_n\}_{n=1}^{\infty}$ tends to the limit $M \cdot L$.

The definition of the limit of a sequence can be written symbolically as:

$$a_n \to L \text{ as } n \to \infty \Leftrightarrow (\forall \epsilon > 0)(\exists n \in \mathbb{N})(\forall m > n)[|a_m - L| < \epsilon]$$

We are asked to prove that:

$$a_n \to L \text{ as } n \to \infty \Leftrightarrow (\forall \epsilon_1 > 0)(\exists n \in \mathbb{N})(\forall m > n)[|M \cdot a_m - M \cdot L| < \epsilon_1]$$

From the definition, we know when the limit conditions are met that:

$$|a_m - L| < \epsilon$$

If we multiply that through by M, we get:

$$|M \cdot a_m - M \cdot L| < M \cdot \epsilon$$

Since we are working with all ϵ, if we set $\epsilon_1 = M \cdot \epsilon$, then we get:

$$(\forall \epsilon_1 > 0)(\exists n \in \mathbb{N})(\forall m > n)[|M \cdot a_m - M \cdot L| < \epsilon_1]$$

This is what we are trying to prove. ■

(9) Given an infinite collection $\boldsymbol{A_n}$, $\boldsymbol{n = 1, 2, \dots, \infty}$ of intervals of the real line, their intersection is defined to be

$$\bigcap_{n=1}^{\infty} \boldsymbol{A_n} = \{\boldsymbol{x} | (\forall \boldsymbol{n})(\boldsymbol{x} \in \boldsymbol{A_n})\}$$

Give an example of a family of intervals $\boldsymbol{A_n}$, $\boldsymbol{n = 1, 2, \dots, \infty}$ such that $\boldsymbol{A_{n+1} \subset A_n}$ for all $\boldsymbol{n}$ and $\bigcap_{n=1}^{\infty} \boldsymbol{A_n} = \emptyset$. Prove that your example has the stated property.

We have an infinite collection of intervals, each of which is supposed to be a proper subset of the previous one. The intersection of a set and its subset is always the subset. Therefore the intersection of all the intervals is simply the last interval in the sequence. When intersection of all the intervals becomes the empty set, the last interval must itself be the empty set. There can be no subsequent intervals because there is no proper subset of the empty set.

I therefore think there is no correct answer to this question. However, I will make two attempts that come close to a correct answer.

First, let us define each interval A_n as the open interval of real numbers $\left(0, \frac{1}{n}\right)$. Each subsequent interval will be a proper subset of the one before it. The completeness property of the real numbers tells us that these intervals will go on forever. Now, if we imagine that infinity actually exists, at that point we will reach the open interval (0,0) and the intersection of all these sets will be the empty set. This definition of a family of intervals therefore has the stated property.

Alternatively, for any natural number m, let us define each interval A_n as the closed interval of real numbers $[0, m - n]$. We will define an interval with its bounds in the wrong order as being empty. After n reaches m, all subsequent intervals, and therefore their intersection, will be the null set.

This is true for any integer m that we choose. We can express this as:

$$(\forall m \in \mathbb{N})[\forall A_n = \{i\}_{i=1}^{m-n}]\left[\bigcap_{n=1}^{\infty} A_n = \emptyset\right]$$

This satisfies the question if we ignore the fact that that there is no proper subset of the null set.

I have provided two tentative answers to this question, one of which violates the meaning of infinity and the other violates the meaning of a proper subset. That is the best I can do.

(10) Give an example of a family of intervals A_n, $n = 1, 2, \ldots, \infty$ such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Here we have the same problem as in question (9). If the intersection of all the sets is a single real number, then the last interval also consists of a single real number. There cannot be any subsequent intervals that are proper subsets, which violates the condition that there are an infinite number of intervals.

I will answer the question by taking a dubious literal interpretation of the concept of infinity.

Define each interval A_n as the closed interval of real numbers $\left[0, \frac{1}{n}\right]$. Each subsequent interval will be a proper subset of the one before it. The completeness property of the real numbers tells us that these intervals will go on forever. Now, given that we are imagining that infinity actually exists, at that point we will reach the closed interval [0,0] and the intersection of all these sets will be the single real number zero. We have therefore met the conditions required in the question – each interval is a subset of the previous interval, and it ends (at infinity) with a single real number.